

# A NOTE ON RADIAL SOLUTIONS OF $\Delta^2 u + u^{-q} = 0$ IN $\mathbb{R}^3$ WITH EXACTLY QUADRATIC GROWTH AT INFINITY

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**ABSTRACT.** Of interest in this note is the following geometric interesting equation  $\Delta^2 u + u^{-q} = 0$  in  $\mathbb{R}^3$ . It was found by Choi–Xu (J. Differential Equations **246**, 216–234) and McKenna–Reichel (Electron. J. Differential Equations **37** (2003)) that the condition  $q > 1$  is necessary and any radially symmetric solution grows at least linearly and at most quadratically at infinity for any  $q > 1$ . In addition, when  $q > 3$  any radially symmetric solution is either exactly linear growth or exactly quadratic growth at infinity. Recently, Guerra (J. Differential Equations **253**, 3147–3157) has shown that the equation always admits a unique radially symmetric solution of exactly given linear growth at infinity for any  $q > 3$  which is also necessary. In this note, by using the phase-space analysis, we show the existence of infinitely many radially symmetric solutions of exactly given quadratic growth at infinity for any  $q > 1$ .

## 1. INTRODUCTION

In this note, we are interested in entire solutions of the following geometric interesting equation

$$\Delta^2 u + u^{-q} = 0 \tag{1.1}$$

in  $\mathbb{R}^3$  with  $q > 0$ . Recently, equations of the type (1.1) have been captured much attention since they are naturally arisen when studying the prescribed Q-curvature problem either in  $\mathbb{R}^3$  (with a flat background metric) or in  $\mathbb{S}^3$ . To be precise, positive smooth solutions of Eq. (1.1) for the case  $q = 7$  correspond to conformal metrics conformally equivalent to the flat metric which have constant Q-curvature in  $\mathbb{R}^3$ . Moreover, upon using the stereographic projection, any conformal metric on  $\mathbb{S}^3$  is simply a suitable pullback of the standard one  $g_{\mathbb{S}^3}$  under the conformal transformation of  $\mathbb{S}^3$  into itself; see [CX09]. For interested readers, we refer to [CX09] and the references therein.

As far as we know, Eq. (1.1) was first studied by Choi and Xu in an preprint in 1999, which is eventually published in [CX09], by Xu in [Xu05], and then by McKenna and Reichel for  $\mathbb{R}^n$  for arbitrary  $n \geq 3$  in [KR03]. To seek for complete conformal metrics on  $\mathbb{S}^3$ , it is often to look for  $C^4$  positive solutions  $u$  of Eq. (1.1) with exactly linear growth at infinity in the sense that  $\lim_{|x| \rightarrow +\infty} u(x)/|x| = \alpha$  for some non-negative constant  $\alpha$  in the case  $q = 7$ . In this scenario, it is worth noticing that,  $C^4$  positive solutions of Eq. (1.1) with exactly linear growth at infinity is completely classified. Indeed, it was found by Choi and Xu that, up to a constant multiple, translation and dilation, there holds  $u(x) = \sqrt{1 + |x|^2}$ . Then, it is natural to study  $C^4$  positive solution  $u$  of Eq. (1.1) when  $q \neq 7$  and when  $u$  is no longer of linear growth at infinity which corresponds to incomplete conformal metrics on  $\mathbb{S}^3$ .

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Date: 7<sup>th</sup> Jul, 2016 at 00:56.

2000 *Mathematics Subject Classification.* 35B45, 35J40, 35J60.

*Key words and phrases.* Biharmonic equation; Negative exponent; Radially symmetry; Phase-space analysis; Quadratic growth at infinity.

As a first step toward answering this question, we first look for radially symmetric solutions of Eq. (1.1). However, in order to understand the motivation of writing this note, we first collect all results found in [CX09] and in [KR03]. The following result is now well-known.

**Theorem 1** (see [CX09, KR03]). *We have the following claims:*

- (a) *If Eq. (1.1) admits a smooth positive solution on  $\mathbb{R}^3$ , then there must hold  $q > 1$ .*
- (b) *If Eq. (1.1) admits a smooth positive solution on  $\mathbb{R}^3$  with exactly linear growth, then  $q > 3$ .*
- (c) *Any radially symmetric solution of Eq. (1.1) grows **at least linearly** at infinity in the sense that  $\liminf_{|x| \rightarrow +\infty} u(x)/|x| > 0$  and **at most quadratically** at infinity in the sense that  $\limsup_{|x| \rightarrow +\infty} u(x)/|x|^{2+\varepsilon} = 0$  for arbitrary  $\varepsilon > 0$ .*
- (d) *If  $1 < q < 3$ , then Eq. (1.1) admits infinitely many radially symmetric, **singular** solution with growth rate **strictly between linear and quadratic**, these are of the form  $br^\beta$  with  $\beta \in (1, 2)$ .*
- (e) *If  $q > 1$ , then there exist radially symmetric and smooth solutions of Eq. (1.1) which grow **super-linearly** at infinity.*
- (f) *If  $q > 3$ , then any radially symmetric and smooth solution of Eq. (1.1) is either **exactly linear growth or exactly quadratic growth** at infinity.*
- (g) *If  $q \geq 7$ , then there exist a unique radially symmetric and smooth solutions of Eq. (1.1) with **linear growth** at infinity.*

Recently, by using the phase-space analysis, Guerra [Gue12] studied the structure of radially symmetric solutions of Eq. (1.1) without assuming  $q = 7$ . As far as we know, he first showed, among others, that Eq. (1.1) also admits solutions with exactly linear growth at infinity for any  $q > 3$ ; see also [Lai14] for another proof based on the variation of parameters formula for ODEs. The following is his finding.

**Theorem 2** (see [Gue12]). *We have the following cases:*

- (a) *For  $q > 3$ , there exists a unique radially symmetric solution of Eq. (1.1) such that  $\lim_{|x| \rightarrow +\infty} u(x)/|x|$  exists.*
- (b) *For  $q = 3$ , there exists a unique radially symmetric solution of Eq. (1.1) such that  $\lim_{|x| \rightarrow +\infty} u(x)/(|x|(\log |x|)^{1/4}) = 2^{1/4}$ .*
- (c) *For  $1 < q < 3$ , there exists a unique radially symmetric solution of Eq. (1.1) such that  $\lim_{|x| \rightarrow +\infty} u(x)/|x|^\tau = K_q^{-1/(q+1)}$  where  $K_q = \tau(2-\tau)(\tau+1)(\tau-1)$  and  $\tau = 4/(q+1)$ .*

In view of Theorem 1(f) and Theorem 2 above, the present note has twofold. First we improve Choi–Xu’s result by showing that there exist radially symmetric solutions of Eq. (1.1) with exactly quadratic growth at infinity. Second, we prove that the quadratic growth can be arbitrary. To be precise, we shall prove the following result.

**Theorem 3.** *Given any  $\kappa > 0$  and any  $q > 1$ , there exist infinitely many radially symmetric solutions  $u$  of exactly quadratic growth at infinity in the sense that*

$$\lim_{|x| \rightarrow +\infty} \frac{u(x)}{|x|^2} = \kappa$$

*holds. Furthermore, the solution  $u$  and the given limit  $\kappa$  are related through the following identity*

$$6\kappa = (\Delta u)(0) - \int_0^{+\infty} tu^{-q}(t)dt.$$

Clearly, an easy consequence of Theorem 3 is that the geometric interesting equation  $\Delta^2 u + u^{-7} = 0$  in  $\mathbb{R}^3$  and its corresponding integral equation  $u(x) = \int_{\mathbb{R}^3} |x-y|u(y)^{-7}dy$  are not equivalent since the latter equation only admits radial solutions with linear growth at infinity; see [Xu05, Theorem 1.1]. Further investigation for the relation between these two equations will be carried out in future. In addition, Theorem 3 shows that at infinity, the highest order term of  $u$  is  $|x|^2$ . In the next result, we study lower order terms of  $u$  at infinity. What we also prove in this note is the following.

**Theorem 4.** *Suppose that  $u$  is a radially symmetric solution with exactly quadratic growth  $\kappa > 0$  at infinity found in Theorem 3 above. Then we have the following further asymptotic behavior:*

(a) For  $q > 3/2$ ,

$$\lim_{|x| \rightarrow +\infty} \frac{u(x) - \kappa|x|^2}{|x|} = \frac{1}{2} \int_0^\infty |x|^2 u^{-q}(x) dx.$$

(b) For  $q = 3/2$ ,

$$\lim_{|x| \rightarrow +\infty} \frac{u(x) - \kappa|x|^2}{|x| \log(|x|)} = \frac{1}{2\kappa^{3/2}}.$$

(c) For  $1 < q < 3/2$ ,

$$\lim_{|x| \rightarrow +\infty} \frac{u(x) - \kappa|x|^2}{|x|^{4-2q}} = \chi,$$

where

$$\chi = \frac{1}{2\kappa^q} \left( \frac{1}{3-2q} - \frac{1}{4-2q} + \frac{1}{3(5-2q)} - \frac{1}{3(2-2q)} \right).$$

To prove Theorems 3 and 4, we closely follow the argument presented in [Gue12]. Before closing this section, it is worth noting that Theorem 3 complements all mentioned results above and hence completes the picture of radially symmetric solutions of (1.1). For clarity, we summary all results above as in Table 1.

$1 < q < 3$	$q = 3$	$3 < q < 7$	$q = 7$	$q > 7$
		<i>necessary if <math>u</math> grows linearly</i>		
			<i><math>u</math> is precise</i>	
		<i><math>u</math> grows either linearly or quadratically</i>		
<i><math>u(r)</math> grows between linear and quadratic</i>				
			$\exists!$ $u$ grows linearly	
		$\exists u$ linearly		
	$\exists u \approx r(\log r)^{\frac{1}{4}}$			
$\exists u \approx r^{4/(1+q)}$				
$\exists$ infinitely many $u$ grows quadratically				

Table 1: Summary of results for radially symmetric solutions  $u(r)$  of (1.1).

(The first three rows are due to Choi–Xu [CX09], the next two rows are due to McKenna–Reichel [KR03], the next three rows are due to Guerra [Gue12], and the last is from Theorem 3.) In one way or another, we know that  $q > 1$  is necessary and radially symmetric solutions of (1.1) must grow linearly up to quadratically. For quadratic growth at infinity, Theorem 3 conclude that there are radially symmetric solutions of (1.1) which have quadratic growth at infinity for all  $q > 1$ .

## 2. PROOF OF THEOREMS 3 AND 4

**2.1. An initial value problem.** The present proof follows the arguments in [Gue12] closely. First, suppose  $\beta > 0$ , we consider the following initial value problem:

$$\begin{cases} \Delta^2 U = -U^{-q}, & U > 0, & r \in (0, R_{\max}(\beta)), \\ U(0) = 1, & U'(0) = 0, & \Delta U(0) = \beta, & (\Delta U)'(0) = 0, \end{cases} \quad (2.1)$$

where  $[0, R_{\max}(\beta))$  is the maximal interval of existence of solutions. (Such an existence of solutions for (2.1) follows from standard ODE theory. A similar problem in  $\mathbb{R}^2$  and  $\mathbb{R}^3$  for  $q = 2$  was studied in [GW08].) The following result, indicating the threshold for  $\beta$ , was obtained in [Gue12, Proposition 2.1].

**Proposition 1.** *Assume that  $q > 1$  and  $\beta > 0$ . Let  $U_\beta$  be the unique local solution of (2.1) above. Then there is a unique  $\beta^* > 0$  such that:*

- (a) *If  $\beta < \beta^*$  then  $R_{\max}(\beta) < \infty$ .*
- (b) *If  $\beta \geq \beta^*$  then  $R_{\max}(\beta) = \infty$ .*
- (c) *If  $\beta \geq \beta^*$  then  $\lim_{r \rightarrow +\infty} \Delta U_\beta(r) \geq 0$ .*
- (d) *We have  $\beta = \beta^*$  if and only if  $\lim_{r \rightarrow +\infty} \Delta U_\beta(r) = 0$ .*

In the rest of our present proof, we set  $u = U_\beta$  for some fixed  $\beta > \beta^*$  but arbitrary. Then it suffices to show that  $u$  has exactly quadratic growth at infinity. The fact that such a limit at infinity can be arbitrary follows from a suitable scaling of  $u$ . As a key step toward this end, we shall study asymptotic behavior of  $u$  in the next subsection.

**2.2. Asymptotic behavior.** To understand the structure of radially symmetric solutions of (1.1), we transform (1.1) into the following system of second order partial differential equations

$$\begin{cases} \Delta u = v & \text{in } \mathbb{R}^3, \\ \Delta v = -u^{-q} & \text{in } \mathbb{R}^3. \end{cases} \quad (2.2)$$

To study the asymptotic behavior of (2.2), we follow the ideas in [HV96]. First, by the Emden–Fowler transformation we set

$$x(t) = \frac{ru'}{u}, \quad y(t) = \frac{rv'}{v}, \quad z(t) = \frac{r^2v}{u}, \quad w(t) = \frac{r^2u^{-q}}{v}, \quad t = \log r. \quad (2.3)$$

Then the system (2.2) is transformed into a 4-dimensional quadratic system of the form

$$\begin{cases} x' = x(-1 - x) + z, \\ y' = y(-1 - y) - w, \\ z' = z(2 - x + y), \\ w' = w(2 - qx - y), \end{cases} \quad (2.4)$$

where  $' = d/dt$ . As indicated in [Gue12], the critical points of (2.4) are

$$\begin{aligned} p_0 &= (0, 0, 0, 0), & p_1 &= (1, -1, 2, 0), & p_2 &= (2, 0, 6, 0), \\ p_3 &= (a, a - 2, a(a + 1), (2 - a)(a - 1)), & p_4 &= (0, 2, 0, -6), & p_5 &= (0, -1, 0, 0), \\ p_6 &= (-1, 0, 0, 0), & p_7 &= (-1, -1, 0, 0), & p_8 &= (-1, q + 2, 0, -(q + 2)(q + 3)), \end{aligned}$$

where  $a = 4/(q+1)$ .

Thanks to Proposition 1, the solution  $u$  of (2.1) exists for all time  $t$ ; hence we can denote

$$\gamma = \lim_{r \rightarrow +\infty} v(r),$$

where, as always, we set  $v = \Delta u$ . By Proposition 1(c, d) we know that  $\gamma > 0$ . Since  $-\Delta v = u^{-q}$  with  $v'(0) = 0$ , we have

$$v(r) = v(0) - \int_0^r t u^{-q} dt + \frac{1}{r} \int_0^r t^2 u^{-q} dt. \quad (2.5)$$

Since  $\gamma > 0$ , there exist two positive constant  $a$  and  $b$  such that  $u(r) \geq ar^2 + b$  holds for all  $r > 0$ . From this, for each  $q > 1$ , we find that  $t^2 u^{-q} \rightarrow 0$  as  $t \rightarrow +\infty$ . Thanks to the l'Hôpital rule, we can pass (2.5) to the limit as  $r \rightarrow +\infty$  to obtain

$$v(0) = \gamma + \int_0^{+\infty} t u^{-q} dt < \infty \quad (2.6)$$

while on the other hand we get

$$v(r) = \gamma + \int_r^{+\infty} t u^{-q} dt + \frac{1}{r} \int_0^r t^2 u^{-q} dt. \quad (2.7)$$

Now, an easy computation leads us to

$$\frac{v(r)}{rv'(r)} = -\frac{r \int_r^{+\infty} t u^{-q} dt + \gamma r}{\int_0^r t^2 u^{-q} dt} - 1.$$

**Claim 1.** There holds  $v(r)/(rv'(r)) \rightarrow -\infty$  as  $r \rightarrow +\infty$ . In other words,  $y(r) \rightarrow 0$  as  $r \rightarrow +\infty$ .

*Proof of Claim 1.* Depending on the value of  $q$ , there are two possible cases.

Case 1. Suppose  $q > 3/2$ , then we immediately see that the integral  $\int_0^{+\infty} t^2 u^{-q} dt$  converges. Hence the claim holds since  $\gamma > 0$ .

Case 2. In this scenario, there holds  $1 < q \leq 3/2$ . Using the l'Hôpital rule, we arrive at

$$\lim_{r \rightarrow +\infty} \frac{v(r)}{rv'(r)} = -\lim_{r \rightarrow +\infty} \frac{\int_r^{+\infty} t u^{-q} dt + \gamma r}{r^2 u^{-q}} = -\infty,$$

thanks to  $u^{-q}/r \rightarrow +\infty$  as  $r \rightarrow +\infty$ .  $\square$

Now using the relation  $\Delta u = v$  with  $u(0) = 1$  and  $u'(0) = 0$ , in a similar fashion of (2.5) we obtain

$$u(r) = 1 + \int_0^r t v dt - \frac{1}{r} \int_0^r t^2 v dt. \quad (2.8)$$

From this, we obtain

$$\frac{u(r)}{r^2 v(r)} = \frac{1}{r^2 v(r)} + \frac{1}{r^2 v(r)} \int_0^r t v dt - \frac{1}{r^3 v(r)} \int_0^r t^2 v dt. \quad (2.9)$$

**Claim 2.** There holds  $u(r)/(r^2 v(r)) \rightarrow 1/6$  as  $r \rightarrow +\infty$ . In other words,  $z(r) \rightarrow 6$  as  $r \rightarrow +\infty$ .

*Proof of Claim 2.* We first use the l'Hôpital rule to estimate the last two terms on the right hand side of (2.9). For the middle term, we clearly have

$$\lim_{r \rightarrow +\infty} \frac{1}{r^2 v(r)} \int_0^r t v(t) dt = \lim_{r \rightarrow +\infty} \frac{1}{2 + rv'(r)/v(r)} = \frac{1}{2},$$

thanks to Claim 1. For the last term, we get

$$\lim_{r \rightarrow +\infty} \frac{1}{r^3 v(r)} \int_0^2 tv(t)dt = \lim_{r \rightarrow +\infty} \frac{1}{3 + rv'(r)/v(r)} = \frac{1}{3}.$$

We are now in a position to estimate  $u(r)/(r^2 v(r))$  when  $r$  is large. Clearly, by (2.9) we know that  $\lim_{r \rightarrow +\infty} u(r)/(r^2 v(r)) = 1/6$  since  $r^2 v(r) \rightarrow +\infty$  as  $r \rightarrow +\infty$ .  $\square$

**Claim 3.** There holds  $u(r)/(ru'(r)) \rightarrow 1/2$  as  $r \rightarrow +\infty$ . In other words,  $x(r) \rightarrow 2$  as  $r \rightarrow +\infty$ .

*Proof of Claim 3.* Using (2.8), we obtain  $u'(r) = r^{-2} \int_0^r t^2 v(t)dt$  which yields

$$\frac{u(r)}{ru'(r)} = \frac{r}{\int_0^r t^2 v(t)dt} + \frac{r \int_0^r tv(t)dt}{\int_0^r t^2 v(t)dt} - 1. \quad (2.10)$$

The l'Hôpital rule applied to the first term on the right hand side of (2.10) gives

$$\lim_{r \rightarrow +\infty} \frac{r}{\int_0^r t^2 v(t)dt} = 0$$

while for the second term we know that

$$\lim_{r \rightarrow +\infty} \frac{r \int_0^r tv(t)dt}{\int_0^r t^2 v(t)dt} = \frac{3}{2}.$$

From this we obtain the desired limit.  $\square$

From Claims 1, 2, and 3 we see that the solutions  $(x, y, z, w)$  corresponding to the radially symmetric solutions with quadratic growth are attracted to the point  $p_2 := (2, 0, 6, 0)$  at infinity. Therefore, the asymptotic behavior is obtained by analyzing the asymptotic behavior of solutions about  $(2, 0, 6, 0)$ .

For  $q > 1$ , we first obtain the linearization of (2.4) at  $(x, y, z, w)$  given by the following matrix

$$\begin{pmatrix} -2x - 1 & 0 & 1 & 0 \\ 0 & -2y - 1 & 0 & 1 \\ -z & z & -x + 2 & 0 \\ -qw & -w & 0 & -qx - y - 2 \end{pmatrix}.$$

At  $p_2$ , this matrix becomes

$$\begin{pmatrix} -5 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ -6 & 6 & 0 & 0 \\ 0 & 0 & 0 & -2q - 2 \end{pmatrix}$$

which has the following eigenvalues:  $\lambda_1 = -1$ ,  $\lambda_2 = -2$ ,  $\lambda_3 = -3$ , and  $\lambda_4 = 2 - 2q$ . Since these eigenvalues are non-zero whenever  $q > 1$ , we conclude that there exists a constant  $c_q \neq 0$  such that the following asymptotic behavior occurs: For  $q > 3/2$

$$\frac{ru'(r)}{u(r)} = 2 + c_q e^{-t} + o(e^{-t}) \quad (2.11)$$

as  $t \rightarrow +\infty$  while for  $q = 3/2$

$$\frac{ru'(r)}{u(r)} = 2 + c_q t e^{-t} + o(t e^{-t}) \quad (2.12)$$

as  $t \rightarrow +\infty$  due to  $\lambda_1 = \lambda_4$ , and for  $1 < q < 3/2$

$$\frac{ru'(r)}{u(r)} = 2 + c_q e^{-(2q-2)t} + o(e^{-(2q-2)t}) \quad (2.13)$$

as  $t \rightarrow +\infty$ .

**2.3. Quadratic growth at infinity can be arbitrary.** We establish in this subsection the fact that if there is some radially symmetric solution  $u$  of (1.1) having

$$\lim_{|x| \rightarrow +\infty} \frac{u(r)}{r^2} = \kappa$$

for some  $\kappa > 0$ , then given any  $\varpi > 0$ , there exists a radially symmetric solution  $v$  of (1.1) such that

$$\lim_{|x| \rightarrow +\infty} \frac{v(r)}{r^2} = \varpi.$$

To see this, we first set

$$v(r) = \left(\frac{\varpi}{\kappa}\right)^\delta u\left(\left(\frac{\varpi}{\kappa}\right)^\alpha r\right).$$

Then it is elementary to see that  $v$  solves (1.1) if  $(1+q)\delta + 4\alpha = 0$ . To fulfill the limit  $\lim_{|x| \rightarrow +\infty} v(r)/r^2 = \varpi$ , it also requires  $\delta + 2\alpha = 1$ . Resolving these conditions for  $\delta$  and  $\alpha$ , we conclude that  $\delta = -2/(q-1)$  and  $\alpha = (q+1)/(2(q-1))$  which are obviously well-defined for all  $q > 1$ .

**2.4. Proof of Theorem 3.** Combining (2.7) and (2.8), we have the following representation

$$u(r) = \frac{r}{2} \int_0^r t^2 u^{-q} dt - \frac{1}{2} \int_0^r t^3 u^{-q} dt + \frac{1}{6r} \int_0^r t^4 u^{-q} dt + \frac{r^2}{6} \int_r^{+\infty} t u^{-q} dt + \frac{\gamma r^2}{6} + 1 \quad (2.14)$$

which is similar to [CX09, Eq. (5.1)]. We note that the representation (2.14) is valid for all  $q > 1$ . In general, the term  $\int_r^{+\infty} t u^{-q} dt$  may not be well-defined for  $q < 2$  if  $u$  solves (1.1).

Note that with  $r = e^t$  we obtain  $h'(t) = ru'(r)/u(r)$  where we set  $h(t) := \log u(r)$ . Therefore, by using (2.11)–(2.13), we obtain

$$h'(t) = \begin{cases} 2 + c_q e^{-t} + o(e^{-t}) & \text{if } q > 3/2 \\ 2 + c_q t e^{-t} + o(t e^{-t}) & \text{if } q = 3/2 \\ 2 + c_q e^{-(2q-2)t} + o(e^{-(2q-2)t}) & \text{if } 1 < q < 3/2 \end{cases}$$

as  $t \rightarrow +\infty$ . Integrating both sides gives

$$\frac{u(r)}{r^2} = \begin{cases} u(1) \exp \int_0^t (c_q e^{-s} + o(e^{-s})) ds & \text{if } q > 3/2 \\ u(1) \exp \int_0^t (c_q s e^{-s} + o(t e^{-s})) ds & \text{if } q = 3/2 \\ u(1) \exp \int_0^t (c_q e^{-(2q-2)s} + o(e^{-(2q-2)s})) ds & \text{if } 1 < q < 3/2 \end{cases}$$

From this, it is easy to see that the following limit  $\lim_{|x| \rightarrow +\infty} u(r)/r^2 = \kappa$  exists for some  $\kappa > 0$ . Now using (2.14) and (2.6), we further obtain

$$\kappa = \frac{\gamma}{6} = \frac{1}{6} \left( (\Delta u)(0) - \int_0^{+\infty} t u^{-q}(t) dt \right) \quad (2.15)$$

as claimed. The fact that  $\kappa$  can be arbitrary follows from the preceding subsection by scaling  $u$ .

Finally, since one can freely choose  $\beta > \beta^*$ , we conclude the existence of infinitely many radially symmetric solutions (1.1) of exactly given quadratic growth at infinity for any  $q > 1$ . To realize this fact, one first pick  $\beta_1 \neq \beta_2 > \beta^*$  and follow the procedure above

to select  $u_{\beta_i}$  with growth  $\kappa_i$  at infinity, that is  $\lim_{|x| \rightarrow +\infty} u_{\beta_i}(r)/r^2 = \kappa_i$  for  $i = 1, 2$ . Then we define

$$w_{\beta_2} = \begin{cases} u_{\beta_2} & \text{if } \kappa_1 = \kappa_2, \\ \left(\frac{\kappa_1}{\kappa_2}\right)^{-\frac{2}{q-1}} u_{\beta_2} \left(\left(\frac{\kappa_1}{\kappa_2}\right)^{\frac{q+1}{2(q-1)}} r\right) & \text{if } \kappa_1 \neq \kappa_2. \end{cases}$$

Note that  $u_{\beta_i}$  are distinct because  $\beta_1 \neq \beta_2$ . Moreover, in the case  $\kappa_1 \neq \kappa_2$  there holds  $w_{\beta_2}(0) = (\kappa_1/\kappa_2)^{-2/(q-1)} \neq 1$ ; hence  $u_{\beta_1} \neq w_{\beta_2}$ . In addition, it is easy to see that  $\lim_{|x| \rightarrow +\infty} w_{\beta_2}(r)/r^2 = \kappa_1$ . Therefore, given  $\varpi > 0$  if we scale  $u_{\beta_1}$  and  $w_{\beta_2}$  to get

$$v_1(r) = \left(\frac{\varpi}{\kappa_1}\right)^{-\frac{2}{q-1}} u_{\beta_1} \left(\left(\frac{\varpi}{\kappa_1}\right)^{\frac{q+1}{2(q-1)}} r\right)$$

and

$$v_2(r) = \left(\frac{\varpi}{\kappa_1}\right)^{-\frac{2}{q-1}} w_{\beta_2} \left(\left(\frac{\varpi}{\kappa_1}\right)^{\frac{q+1}{2(q-1)}} r\right),$$

it is immediate to see that  $v_1 \neq v_2$  and that

$$\lim_{|x| \rightarrow +\infty} \frac{v_1(r)}{r^2} = \lim_{|x| \rightarrow +\infty} \frac{v_2(r)}{r^2} = \varpi.$$

The proof is complete.

**2.5. Proof of Theorem 4.** Let  $u$  be a solution of (1.1) constructed as above which has quadratic growth  $\kappa$  at infinity. Using the quadratic growth formula (2.15) and the presentation (2.14), we know that the constructed solution  $u$  also fulfills following presentation

$$u(r) - \kappa r^2 = \frac{r}{2} \int_0^r t^2 u^{-q} dt - \frac{1}{2} \int_0^r t^3 u^{-q} dt + \frac{1}{6r} \int_0^r t^4 u^{-q} dt + \frac{r^2}{6} \int_r^{+\infty} t u^{-q} dt + u(0), \quad (2.16)$$

and this presentation is also valid for all  $q > 1$ , compared with [CX09, Eq. (5.1)].

Then we make use of (2.11)–(2.13) plus the l'Hôpital rule to conclude the theorem. For example, when  $q > 3/2$ , there holds  $(u(r) - \kappa r^2)/r \rightarrow \int_0^{+\infty} t^2 u^{-q} dt$  due to the contribution of the first integral in (2.16) since  $r^3 u^{-q}(r) \rightarrow 0$  as  $r \rightarrow +\infty$ . When  $q = 3/2$ ,  $(u(r) - \kappa r^2)/(r \log r) \rightarrow 1/(2\kappa^{3/2})$  due to the contribution of the second integral in (2.16) while in the case  $q < 3/2$ ,  $(u(r) - \kappa r^2)/r^{4-2q} \rightarrow \chi$  due to the contribution of all four integrals in (2.16). This completes our proof of Theorem 4.

#### ACKNOWLEDGMENTS

We thank Prof. Ignacio Guerra for useful discussion about his paper [Gue12]. The second author was partially supported by Vietnam National University at Hanoi through VNU Scientist Links.

#### REFERENCES

- [CX09] Y.S. CHOI, X. XU, Nonlinear biharmonic equations with negative exponents, *J. Differential Equations* **246** (2009), pp. 216–234. [1](#), [2](#), [3](#), [7](#), [8](#)
- [Gue12] I. GUERRA, A note on nonlinear biharmonic equations with negative exponents, *J. Differential Equations* **253** (2012), pp. 3147–3157. [2](#), [3](#), [4](#), [8](#)
- [GW08] Z.M. GUO, J.C. WEI, Entire solutions and global bifurcations for a biharmonic equation with singular non-linearity in  $\mathbb{R}^3$ , *Adv. Differential Equations* **13** (2008), pp. 753–780. [4](#)
- [HV96] J. HULSHOF, R.C.A.M. VAN DER VORST, Asymptotic behaviour of ground states, *Proc. Amer. Math. Soc.* **124** (1996), pp. 2423–2431. [4](#)
- [KR03] P.J. MCKENNA, W. REICHEL, Radial solutions of singular nonlinear biharmonic equations and applications to conformal geometry, *Electron. J. Differential Equations* **37** (2003), pp. 1–13. [1](#), [2](#), [3](#)
- [Lai14] B. LAI, A new proof of I. Guerra's results concerning nonlinear biharmonic equations with negative exponents, *J. Math. Anal. Appl.* **418** (2014), pp. 469–475. [2](#)



- [Xu05] X. Xu, Exact solutions of nonlinear conformally invariant integral equations in  $\mathbf{R}^3$ , *Adv. Math.* **194** (2005), pp. 485–503. [1](#), [3](#)

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